

# Desirable Decompositions of Generalized Nevanlinna Functions

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## Abstract

For a given generalized Nevanlinna function  $Q \in N_\kappa(H)$ , we study decompositions that satisfy:  $Q = Q_1 + Q_2$ ;  $Q_i \in N_{\kappa_i}(H)$ , and  $\kappa_1 + \kappa_2 = \kappa$ ,  $0 \leq \kappa_i$ , which we call desirable decompositions. In this paper, some sufficient conditions for such decompositions of  $Q$  are given.

One of the main results is a new operator representation of  $\hat{Q}(z) := -Q(z)^{-1}$  if  $Q(z) := \Gamma_0^+(A - z)^{-1}\Gamma_0$ , where  $A$  is a bounded self-adjoint operator in a Pontryagin space. The new representation is used to get an interesting desirable decomposition of  $\hat{Q}$  and to obtain some information about singularities of  $\hat{Q}$ .

**Key words:** Generalized Nevanlinna functions, Operator representations, Pontryagin space

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## 1 Preliminaries and introduction

**1.1. Preliminaries.** Let  $N_0$ ,  $R$ ,  $C$  denote the sets of non-negative integers, real numbers, and complex numbers, respectively. Let  $(\cdot, \cdot)$  denote a definite scalar product in a Hilbert space  $H$  and let us denote by  $L(H)$  the space of bounded linear operators in  $H$ .

**Definition 1.1** *An operator valued complex function  $Q : D(Q) \rightarrow L(H)$  belongs to the class of **generalized Nevanlinna functions**  $N_\kappa(H)$  if it satisfies the following requirements:*

- $Q$  is meromorphic in  $C \setminus R$ ,
- $Q(\bar{z})^* = Q(z)$ ,  $z \in D(Q)$ ,
- the Nevanlinna kernel  $N_Q(z, \omega) := \frac{Q(z) - Q(\omega)^*}{z - \bar{\omega}}$ ;  $z, \omega \in D(Q) \cap \mathbf{C}^+$ ,

*has  $\kappa$  negative squares, i.e. for arbitrary  $n \in N_0$ ,  $z_1, \dots, z_n \in D(Q) \cap \mathbf{C}^+$  and  $h_1, \dots, h_n \in H$  the Hermitian matrix  $(N_Q(z_i, z_j)h_i, h_j)_{i,j=1}^n$  has at most  $\kappa$  negative eigenvalues, and for at least one choice of  $n$ ;  $z_1, \dots, z_n$ , and  $h_1, \dots, h_n$  it has exactly  $\kappa$  negative eigenvalues.*

Let  $\kappa \in N_0$  and let  $(K, [\cdot, \cdot])$  denote a Krein space. If the scalar product  $[\cdot, \cdot]$  has  $\kappa (< \infty)$  negative squares it is called a Pontryagin space of index  $\kappa$ . The definition of a Pontryagin space and other concepts related to it can be found e.g. in [6].

For a bounded linear operator  $\Gamma : H \rightarrow K$ , we denote by  $\Gamma^+ : K \rightarrow H$  the operator defined by  $(h, \Gamma^+k) := [\Gamma h, k]$ ,  $h \in H$ ,  $k \in K$ .

We will deal with the following characterization of generalized Nevanlinna functions rather than with Definition 1.1.

**Proposition 1.2** *A function  $Q : D(Q) \rightarrow L(H)$  is a generalized Nevanlinna function of index  $\kappa$ , denoted by  $Q \in N_\kappa(H)$ , if and only if it has the representation of the form*

$$Q(z) = Q(z_0)^* + (z - \bar{z}_0)\Gamma^+ \left( I + (z - z_0)(A - z)^{-1} \right) \Gamma, \quad z \in D(Q) \quad (1.1)$$

where,  $A$  is a self-adjoint linear relation in some Pontryagin space  $(K, [.,.])$  of index  $\tilde{\kappa} \geq \kappa$ ;  $\Gamma : H \rightarrow K$  is a bounded operator;  $z_0 \in \rho(A) \cap \mathbf{C}^+$  is a fixed point of reference. This representation can be chosen to be minimal, that is

$$K = \text{cls} \left\{ \left( I + (z - z_0)(A - z)^{-1} \right) \Gamma h : z \in \rho(A), h \in H \right\}.$$

The representation is minimal if and only if the negative index of the Pontryagin space  $\tilde{\kappa}$  equals  $\kappa$ . In that case the triplet  $(K, A, \Gamma)$  is uniquely determined (up to isomorphism) and we say that  $A$  and  $\Gamma$  are closely connected.

Note that in the special case when “negative index”  $\kappa = 0$ , the Pontryagin space reduces to a Hilbert space.

Such operator representations were developed by M. G. Krein and H. Langer, see e.g. [7, 8] and later translated to representations in terms of linear relations (multivalued operators), see e.g. [5].

In this paper, a point  $\alpha \in C$  is called a generalized pole of  $Q$  if it is an eigenvalue of the representing relation  $A$  of the function  $Q$  given by (1.1). It means that it may be isolated (i.e. an ordinary pole) as well as an embedded singularity of  $Q$ .

For later reference we collect the following well known facts into a lemma.

**Lemma 1.3** *If  $Q \in N_\kappa(H)$  is represented by (1.1) then it holds*

$$Q(z) = Q_\alpha(z) + H_\alpha(z), \quad (1.2)$$

where  $Q_\alpha \in N_{\kappa_1}(H)$ ,  $H_\alpha \in N_{\kappa_2}(H)$  is a holomorphic function at  $\alpha$ ,  $\kappa_1 + \kappa_2 = \kappa$ . One can always select  $Q_\alpha$  to be holomorphic at  $\infty$ . Then  $Q_\alpha$  admits the representation

$$Q_\alpha(z) = \Gamma_0^+ (A_0 - z)^{-1} \Gamma_0, \quad (1.3)$$

with a bounded operator  $A_0$ .

If  $\Gamma_0^+ \Gamma_0$  is not boundedly invertible, one can add a convenient function to  $Q_\alpha$  in (1.2) and subtract the same function from  $H_\alpha$  so that the new function  $Q'_\alpha = \Gamma_0'^+ (A'_0 - z)^{-1} \Gamma_0'$  has the same negative index  $\kappa_1$ ,  $\Gamma_0'^+ \Gamma_0'$  is boundedly invertible and  $A'_0 \supseteq A_0$ . Then the negative index of  $H'_\alpha$  is  $\geq \kappa_2$ .

By returning to the previous notation, one can consider that  $\Gamma_0^+ \Gamma_0$  is boundedly invertible in (1.3) and now it only holds  $\kappa_1 + \kappa_2 \geq \kappa$  in (1.2).

In either case, if  $\alpha \in R$  is a generalized pole of  $Q$ , then the operator  $A_0$  that represents  $Q_\alpha$  has the same root manifold at  $\alpha$  as relation  $A$ .

Because of those properties of  $Q_\alpha$ , it is not a loss of generality if one deals with  $Q_\alpha$  rather than with  $Q$  when researching properties of  $Q$  at  $\alpha$ .

Recall here the following statement, see [9, 10], which we will also use for later references.

**Lemma 1.4** *Let the function  $Q \in N_\kappa(H)$  has minimal representation (1.1). If  $Q(z_0)$  is boundedly invertible then the inverse function  $\hat{Q}(z) := -Q(z)^{-1}$  belongs to the class  $N_\kappa(H)$  and admits the minimal representation*

$$\hat{Q}(z) = -Q(\bar{z}_0)^{-1} + (z - \bar{z}_0)\hat{\Gamma}^+ (I + (z - z_0)(\hat{A} - z)^{-1})\hat{\Gamma}, \quad (1.4)$$

where  $\hat{\Gamma} = -\Gamma Q(z_0)^{-1}$ . Moreover, for  $z \in \rho(A) \cap \rho(\hat{A})$  and

$$\Gamma_z = (I + (z - z_0)(A - z)^{-1})\Gamma \quad (1.5)$$

it holds

$$(\hat{A} - z)^{-1} - (A - z)^{-1} = -\Gamma_z Q(z)^{-1} \Gamma_z^+. \quad (1.6)$$

**1.2. Introduction.** When one studies a complicated object, a way to go is to break it down to simpler components. The same is true with generalized Nevanlinna functions. Various breakdowns of those functions have been proven; some additive (decompositions), some multiplicative (factorizations). In this paper, we will focus on decompositions.

It is well known that a sum  $Q$  of generalized Nevanlinna functions that satisfies

- (a)  $Q_i \in N_{\kappa_i}(H)$ ,  $0 \leq \kappa_i$ ,  $i = 1, 2$ ,
- (b)  $Q(z) = Q_1(z) + Q_2(z)$ ,

belongs to some generalized Nevanlinna class  $N_\kappa(H)$  and that it holds  $\kappa_1 + \kappa_2 \geq \kappa$ . However, the decompositions with  $\kappa_1 + \kappa_2 > \kappa$  are not particularly interesting because then the properties of component functions  $Q_i$  do not add up correctly to the properties of  $Q$ . In this paper, our main goal is to find necessary and sufficient conditions that functions  $Q_1$  and  $Q_2$  satisfying conditions (a) and (b) also satisfy

- (c)  $\kappa_1 + \kappa_2 = \kappa$ .

A decomposition that satisfy (a), (b) and (c) we call a *desirable decomposition*. Obviously, that definition can be extended to the sums of more than two functions.

Some sufficient conditions that a function satisfying (a) and (b) also satisfies (c) were given for scalar functions in [8] and for matrix functions in [4], Proposition 3.2. However, those papers only dealt with functions  $Q_i \in N_{\kappa_i}^{n \times n}$ ,  $i = 1, 2$  that have disjoint sets of generalized poles not of positive type, i.e.  $\sigma_0(Q_1) \cap \sigma_0(Q_2) = \emptyset$ . In addition, it was assumed: If  $\alpha \in \sigma_0(Q_j) \cap R$ , then  $\lim_{\eta \rightarrow 0} \eta Q_k(\alpha + i\eta) = 0$  and if  $\infty \in \sigma_0(Q_j)$ , then  $\lim_{\eta \rightarrow \infty} \eta^{-1} Q_k(i\eta) = 0$ ,  $j \neq k$ ,  $j, k = 1, 2$ .

In Section 2, Theorem 2.3, we give some sufficient conditions for desirable decompositions in the most general case, for functions of the form (1.1). In addition to that, for a given functions  $Q_i$  that satisfy (a) we give sufficient conditions that the sum  $Q = Q_1 + Q_2$  belongs  $N_{\kappa_1 + \kappa_2}(H)$ , which means that the number of negative squares is preserved. In order to do that we had to introduce the following assumption

- (d)  $\Gamma^+$  is injection.

Our results also apply to desirable decompositions of a given function  $Q$  where components  $Q_i$  have the same critical point. A decomposition where the decomposing functions have a common critical point we call a decomposition *within a critical point*.

Example 2.5 is complementary to the statements 2.1 through 2.4 because it explains assumptions of those statements. It also shows us that converse statement of Theorem 2.3 (i) does not hold.

In the short Section 3, we derive one desirable decomposition within a generalized pole  $\alpha$  in terms of the maximal Jordan chains in that pole.

In Section 4, the main result is Theorem 4.2. In that theorem we assume  $Q(z) := \Gamma_0^+(A - z)^{-1} \Gamma_0 \in N_\kappa(H)$ , where  $A$  is a bounded self-adjoint operator in a Pontryagin space and  $\Gamma_0^+ \Gamma_0$  is boundedly invertible and we prove a new, operator representation of  $\hat{Q}(z) := -Q(z)^{-1}$ .

In Section 5, we use that representation of  $\hat{Q}$  to find the decomposition

$$\hat{Q}(z) = \hat{Q}_1(z) + \hat{Q}_2(z); \quad \hat{Q}_1 \in N_{\hat{\kappa}_1}(H), \quad \hat{Q}_2 \in N_{\hat{\kappa}_2}(H); \quad \hat{\kappa}_1 + \hat{\kappa}_2 = \kappa,$$

where one of the components, e.g.  $\hat{Q}_1$  can have only a zero in the critical point  $\alpha$  of  $\hat{Q}$ .

## 2 Desirable decomposition of a generalized Nevanlinna function

**2.1.** According to Lemma 1.3, representations of the form (1.3) play an important role in decompositions of generalized Nevanlinna functions. To simplify notation, we will deal with a function  $Q \in N_\kappa(H)$

$$Q(z) = \Gamma_0^+ (A - z)^{-1} \Gamma_0, \quad (2.1)$$

where  $\Gamma_0 : H \rightarrow K$  is a bounded operator and  $A$  is a bounded self-adjoint operator in a Pontryagin space  $K$ . We will always denote by  $\Gamma$  operator used in representation (1.1) and by  $\Gamma_0$  operator used in the special case, representation (2.1).

**Lemma 2.1** (i) *If  $\Gamma^+$  in representation (1.1) is an injection, then (1.1) is minimal representation of  $Q$ .*

(ii) *If  $\Gamma_0^+ \Gamma_0$  is injection, then the function  $Q$  given by (2.1) satisfies*

$$(f, Q(z)h) = 0, \forall z \in D(Q), \forall h \in H \rightarrow f = 0 \quad (2.2)$$

(iii) *If (2.2) holds then  $\Gamma_0$  is injection.*

(iv) *Assume (2.1) is minimal. If  $\Gamma_0$  is injection then (2.2) holds.*

**Proof.** (i). Assume  $\Gamma^+$  is an injection and

$$0 = \left[ y, \left( I + (z - z_0)(A - z)^{-1} \right) \Gamma h \right], \forall z \in \rho(A), \forall h \in H$$

Then

$$\left( \Gamma^+ \left( I + (\bar{z} - \bar{z}_0)(A - \bar{z})^{-1} \right) y, h \right) = 0, \forall z \in \rho(A), \forall h \in H$$

As  $\Gamma^+$  is injection, we conclude

$$\left( I + (\bar{z} - \bar{z}_0)(A - \bar{z})^{-1} \right) y = 0 \quad (2.3)$$

The obvious solution of the equation (2.3) is  $y = 0$ . If that is the only solution of (2.3) then minimality of the representation (1.1) is proven. Assume to the contrary that equation (2.3) has a nonzero solution  $y$ . Then it follows:

As  $\bar{z} \in \rho(A)$ , the relation  $(A - \bar{z})^{-1}$  is defined on  $K$ . That implies,

$$(A - \bar{z})(A - \bar{z})^{-1} \supseteq I$$

Then from (2.3) we have  $-(\bar{z} - \bar{z}_0)y \in (A - \bar{z})y$  and therefore  $\bar{z}_0 y \in Ay$ . This means that  $\bar{z}_0$  is an eigenvalue of  $A$ . It is in contradiction with the fact that  $\bar{z}_0$  is regular point of  $A$  as the symmetrical point of  $z_0 \in C^+ \cap \rho(A)$ . Therefore, it has to be  $y = 0$ , which proves that representation (1.1) is minimal.

(ii)  $0 = (f, Q(z)h), \forall z \in D(Q), \forall h \in H$

$$\rightarrow 0 = (f, zQ(z)h), \forall z \in D(Q), \forall h \in H$$

$$\rightarrow 0 = \lim_{z \rightarrow \infty} (f, zQ(z)h) = - (f, \Gamma_0^+ \Gamma_0 h) = - (\Gamma_0^+ \Gamma_0 f, h), \forall h \in H.$$

As  $H$  is a Hilbert space and  $\Gamma_0^+ \Gamma_0$  is injection, it follows  $f = 0$ .

(iii) Assume that (2.2) holds and  $\Gamma_0 f = 0$ . Then,

$$0 = [\Gamma_0 f, \Gamma_0 h] = \left[ \Gamma_0 f, (A - z)^{-1} \Gamma_0 h \right] = (f, Q(z)h), \forall z \in D(Q), \forall h \in H.$$

Now from (2.2) it follows  $f = 0$ , which proves that  $\Gamma_0$  is injection.

By similar arguments the statement (iv) can be proven.

Note that the representation (2.1) can always be selected to be minimal. Then, the statements (iii) and (iv) mean that (2.2) holds if and only if  $\Gamma_0$  is injection.

The converse statement of (iv) does not hold. In the Example 2.5 we will see that it is possible to have (2.2) satisfied and  $\Gamma_0$  is injection, but the corresponding representation (2.1) does not need to be minimal.

**2.2.** Let us now assume that functions  $Q_i \in N_{\kappa_i}(H)$  are of the form (1.1) represented by triplets  $(K_i, A_i, \Gamma_i)$ ,  $i = 1, 2$ , i.e.

$$Q_i(z) = Q_i(z_0)^* + (z - \bar{z}_0)\Gamma_i^+ \left( I_i + (z - z_0)(A_i - z)^{-1} \right) \Gamma_i \quad (2.4)$$

where  $A_i$  are self-adjoint relations and  $\Gamma_i : H \rightarrow K_i$ . Then we can introduce the triplet  $(\tilde{K}, A, \Gamma)$  by:

$$\tilde{K} := K_1 [+] K_2,$$

$$A := \{ \{k_1 [+] k_2, h_1 [+] h_2\} : \{k_i, h_i\} \in A_i, i = 1, 2 \} \quad (2.5)$$

Because  $A_i$  are self-adjoint in  $K_i$ ,  $A$  is self-adjoint in  $K_1 [+] K_2$ , i.e.  $A = A^{[*]}$ .

$$\Gamma : H \rightarrow K_1 [+] K_2, \Gamma(h) := \Gamma_1(h) [+] \Gamma_2(h), \Gamma_i(h) \in K_i \quad (2.6)$$

Then  $\Gamma^+ = \Gamma_1^+ + \Gamma_2^+ : K_1 [+] K_2 \rightarrow H$  is defined by

$$[\Gamma h, k_1 [+] k_2] = (h, \Gamma_1^+ k_1 + \Gamma_2^+ k_2) \quad (2.7)$$

where  $k_i \in K_i$ .

The same definitions hold when functions  $Q_i$  are of the form (2.1) with  $\Gamma_{0i}$  and  $\Gamma_0$  replacing  $\Gamma_i$  and  $\Gamma$ . The above definitions prepared us for the following lemma.

**Lemma 2.2** *Let functions  $Q_i$  be represented by (2.4). For the function  $Q := Q_1 + Q_2$ , let us observe the representation*

$$Q(z) = Q_1(z_0)^* + Q_2(z_0)^* + (z - \bar{z}_0)\Gamma^+ \begin{pmatrix} I_1 + (z - z_0)(A_1 - z)^{-1} & 0 \\ 0 & I_2 + (z - z_0)(A_2 - z)^{-1} \end{pmatrix} \Gamma \quad (2.8)$$

where functions  $\Gamma$  and  $\Gamma^+$  are defined by (2.6), (2.7), respectively.

(i) *If operator  $\Gamma^+ : K_1 [+] K_2 \rightarrow H$  introduced by (2.7) is injection, then (2.8) is minimal.*

(ii) *Let now functions  $Q_i$ ,  $i = 1, 2$  be given by (2.1), let  $\Gamma_0$  and  $\Gamma_0^+$  be defined by (2.6), (2.7) and let representation*

$$Q(z) := Q_1(z) + Q_2(z) = \Gamma_0^+ \begin{pmatrix} (A_1 - z)^{-1} & 0 \\ 0 & (A_2 - z)^{-1} \end{pmatrix} \Gamma_0 \quad (2.9)$$

*be minimal. If at least one of  $\Gamma_{0i}$  is an injection then it holds*

$$(f, Q(z)h) = 0, \forall z \in D(Q), \forall h \in H \rightarrow f = 0$$

**Proof.** (i). The statement follows when we apply Lemma 2.1 (i) to function  $Q$  represented by (2.8), where  $K$ ,  $A$ ,  $\Gamma$  and  $\Gamma^+$  are defined by (2.5), (2.6) and (2.7).

(ii) Assume now

$$(f, Q(z)h) = (f, Q_1(z)h) + (f, Q_2(z)h) = 0, \forall z \in D(Q), \forall h \in H$$

Then

$$\left[ \Gamma_{01} f, (A_1 - z)^{-1} \Gamma_{01} h \right] + \left[ \Gamma_{02} f, (A_2 - z)^{-1} \Gamma_{02} h \right] = 0, \forall z \in \rho(A), \forall h \in H$$

From assumption of minimality of  $Q$  we conclude

$$\begin{pmatrix} \Gamma_{01} f \\ \Gamma_{02} f \end{pmatrix} = 0$$

From the assumption that at least one  $\Gamma_{0i}$  is injection we have  $f = 0$ .

Note, we did **not** assume minimality of individual representations (2.4). Minimality of components does not guarantee the minimality of the sum (2.8), while the minimality of the sum guaranties minimality of the components in the representations (2.8) and (2.9).

**Theorem 2.3** (i) Assume representation (1.1) of  $Q \in N_\kappa(H)$  is minimal, and there exist non-degenerate, invariant with respect to  $A$  sub-spaces  $K_i, K_1 [+] K_2 = K$ . Then

(a)  $\exists Q_i \in N_{\kappa_i}(H)$ , minimally represented by triplets  $(K_i, A_i, \Gamma_i)$ ,

(b)  $Q(z) = Q_1(z) + Q_2(z)$ ,

(c)  $\kappa_1 + \kappa_2 = \kappa$ .

(ii) If conditions (a), (b) and

(d)  $\Gamma^+ = \Gamma_1^+ + \Gamma_2^+$  is an injection;

are satisfied, then the unique minimal representation of  $Q$  is given by (2.8), where the representing triplet  $(K, A, \Gamma)$  is defined by (2.5), (2.6) and (2.7). In addition,  $Q \in N_{\kappa_1 + \kappa_2}(H)$ , i.e. (c) holds.

**Proof (i)** We will prove the proposition under seemingly more general assumptions. We will assume existence of only one non-degenerate invariant subspace  $K_1 \subseteq K$ . Then we introduce the orthogonal projection onto  $K_1$ ,  $E_1 : K \rightarrow K_1$ . For  $E_2 := I - E_1$  and  $K_2 := E_2 K$  the following decomposition holds

$$K = K_1 [+] K_2,$$

where  $K_1$  and  $K_2$  are Pontryagin subspaces of negative indexes  $\kappa_i, 0 \leq \kappa_i, i = 1, 2$ ,  $\kappa_1 + \kappa_2 = \kappa$ .

Because  $A$  is a self-adjoint relation,  $K_2$  is also invariant with respect to  $A$ . Then

$$\begin{aligned} & I + (z - z_0)(A - z)^{-1} = \\ & = \begin{pmatrix} E_1 + (z - z_0)[E_1(A - zI)E_1]^{-1} & 0 \\ 0 & E_2 + (z - z_0)[E_2(A - zI)E_2]^{-1} \end{pmatrix}. \end{aligned} \quad (2.10)$$

If we introduce  $A_i := E_i A E_i : K_i \rightarrow K_i, \Gamma_i := E_i \Gamma, i = 1, 2$ , then it holds

$$Q(z) = Q_1(z) + Q_2(z),$$

where

$$Q_i(z) := Q_i(z_0)^* + (z - \bar{z}_0) \Gamma_i^+ \left( I + (z - z_0) (A_i - z)^{-1} \right) \Gamma_i. \quad (2.11)$$

From (2.10) and from the minimality of the representation (1.1), the minimality of representations (2.11) follows.

Indeed, for  $y_1[+]y_2 \in K_1[+]K_2$  minimality of (1.1) means

$$\begin{aligned} & \left[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} E_1 + (z - z_0) [E_1 (A - zI) E_1]^{-1} & 0 \\ 0 & E_2 + (z - z_0) [E_2 (A - zI) E_2]^{-1} \end{pmatrix} \begin{pmatrix} \Gamma_1 h \\ \Gamma_2 h \end{pmatrix} \right] \\ & = 0, \forall z \in \rho(A), \forall h \in H, \rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \end{aligned}$$

If we keep  $y_2 = 0$ , we conclude that  $Q_1$  is minimally represented by  $(K_1, A_1, \Gamma_1)$ . By the same token we conclude that  $Q_2$  is minimally represented by  $(K_2, A_2, \Gamma_2)$ .

It further means that negative indexes of  $Q_i$  and  $K_i$  are equal. Hence,  $Q_i \in N_{\kappa_i}(H)$ ,  $i = 1, 2$  and from  $K = K_1[+]K_2$  we get  $\kappa_1 + \kappa_2 = \kappa$ .

That proves (a), (b) and (c).

**Proof (ii)** Assume now that functions  $Q_i$ , represented by (1.1), satisfy conditions (a), (b) and (d). According to Lemma 2.2 (i), the representation (2.8) is minimal representation in terms of the triplet  $(\tilde{K}, A, \Gamma)$  defined by (2.5), (2.6) and (2.7). The minimality of representations of  $Q_i$  implies that negative indexes of  $K_i$  are  $\kappa_i$ , respectively. From  $\tilde{K} = K_1[+]K_2$  it follows that negative index  $\tilde{\kappa}$  of  $\tilde{K}$  satisfies  $\tilde{\kappa} = \kappa_1 + \kappa_2$ .

From the minimality of the representation (2.8) of  $Q$  in terms of  $(\tilde{K}, A, \Gamma)$  it follows  $Q \in N_{\tilde{\kappa}}(H)$ . Hence,  $Q \in N_{\kappa_1 + \kappa_2}(H)$ .

Then  $K_1 \subseteq \tilde{K}$  is the non-degenerate subspace invariant with respect to relation  $A$ .

From the uniqueness (up to isomorphism) of the representing triplet of the minimal representation, we conclude that representation (1.1) is of the form (2.8), and we can denote  $\tilde{K}$  by  $K$ . This proves the statement (ii).

By means of the triplet  $(\tilde{K}, A, \tilde{\Gamma})$ , where  $\tilde{K}$  and  $A$  are as before and  $\tilde{\Gamma} : H_1(+)H_2 \rightarrow K_1[+]K_2$  is defined by

$$\tilde{\Gamma}(h_1(+)h_2) := \Gamma_1 h_1[+] \Gamma_2 h_2$$

it is easy to prove the following proposition.

**Proposition 2.4** *If  $Q_i \in N_{\kappa_i}(H_i)$ ,  $i = 1, 2$ , and*

$$\tilde{Q}(z) := \begin{pmatrix} Q_1(z) & 0 \\ 0 & Q_2(z) \end{pmatrix},$$

*then  $\tilde{Q} \in N_{\kappa_1 + \kappa_2}(H_1(+)H_2)$ .*

**2.3.** The following example explains many assumptions of the statements 2.1 through 2.4, making them natural.

**Example 2.5** *Given the matrix function*

$$Q(z) = - \begin{pmatrix} z^{-1} + z^{-2} & z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}.$$

Obviously  $Q$  is a regular (boundedly invertible for every  $z \neq 0$ ) function of the form (2.1) but  $Q'(\infty) = -\Gamma_0^+ \Gamma_0$  is not even an injection.

$$Q(z) = - \begin{pmatrix} z^{-1} + z^{-2} & z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}$$

$$= - \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} - \begin{pmatrix} z^{-2} & z^{-1} \\ z^{-1} & 0 \end{pmatrix} =: Q_1(z) + Q_2(z).$$

Because the scalar function  $q(z) = -z^{-1}$  belongs to class  $N_0$ , according to Proposition 2.4 it holds

$$Q_1(z) := - \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} = \Gamma_1^+ (A_1 - z)^{-1} \Gamma_1 \in N_0^{2 \times 2},$$

where  $\Gamma_1 = \Gamma_1^+ = \Gamma_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ .

$$Q_2(z) := - \begin{pmatrix} z^{-2} & z^{-1} \\ z^{-1} & 0 \end{pmatrix} = \Gamma_2^+ (A_2 - z)^{-1} \Gamma_2,$$

where  $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\Gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

It is easy to see that both functions  $Q_i$  are minimally represented.

Formula (2.9) gives here

$$\begin{aligned} Q_1(z) + Q_2(z) &= (\Gamma_1^+ \ \Gamma_2^+) \begin{pmatrix} (A_1 - z)^{-1} & 0 \\ 0 & (A_2 - z)^{-1} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \\ &= - \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} z^{-1} & z^{-2} \\ 0 & z^{-1} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \\ &= - \begin{pmatrix} z^{-1} + z^{-2} & z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}. \end{aligned} \tag{2.12}$$

Let  $(\tilde{K}, A, \Gamma_0)$  be the triplet created by means formulae (2.5), (2.6) and (2.7).

$$\begin{aligned} J &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Gamma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_0^+ = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Obviously,  $\Gamma_0$  is injection but  $\Gamma_0^+$  and  $\Gamma_0^+ \Gamma_0$  are not. Assume now

$$\left( \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} z^{-1} + z^{-2} & z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right) = 0, \quad \forall h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbf{C}^2$$

As the functions  $g_1(z) := (z^{-1} + z^{-2})h_1 + z^{-1}h_2$  and  $g_2(z) := z^{-1}h_1 + z^{-1}h_2$  are obviously linearly independent for every fixed  $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ , it has to be  $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$ .

This means that (2.2) holds and  $\Gamma_0$  is injection. However, it is easy to verify that the corresponding representation (2.12) of  $Q$  is not minimal. Hence, converse statements of Lemma 2.1 (iv) and Lemma 2.2 (ii) do not hold.

Note that conditions (a), (b) and (c) are satisfied in this example but it is not sufficient for minimality of (2.12). That means that the converse statement of Theorem 2.3 (i) does not hold. It justifies introduction of the condition (d) in the study of desirable decompositions. Note also that minimal representation of  $Q$  must be different form (2.12).



### 3 Decomposition of the Pontryagin space by means of Jordan chains of a self-adjoint relation

**3.1.** Let us denote root manifold (algebraic eigenspace) of the representing relation  $A$  at  $\alpha \in R$  by  $S_\alpha(A) := \{x : \exists r \in N, (A - \alpha)^r x = 0\}$ .

Let  $X = \{x_k, k = 0, \dots, l-1\}$  be a maximal non-degenerate Jordan chain at  $\alpha$  of the representing relation  $A$  of  $Q \in N_\kappa(H)$ . According to Lemma 1.3,  $X$  is also Jordan chain of the bounded self-adjoint operator  $A_0$  representing the function  $Q_\alpha$ .

**Proposition 3.1** *Let  $Q \in N_\kappa(H)$  be given by minimal representation (1.1) and let  $X$  be a maximal non-degenerate Jordan chain of the length  $l$  of the representing relation  $A$  at  $\alpha$ . Then*

$$(i) \quad Q(z) = Q_1(z) + Q_2(z),$$

$$\text{where } Q_i \in N_{\kappa_i}(H), i = 1, 2; \kappa_1 + \kappa_2 = \kappa; \sigma(Q_1) = \{\alpha\} \text{ and } Q_1(z) = \Gamma_1^+ (A_1 - z)^{-1} \Gamma_1.$$

(ii) *There exist  $x_{l-1} \in \Gamma_1(H)$ , such that*

$$X = \left\{ (A - \alpha)^{l-1-k} x_{l-1}, k = 0, \dots, l-1 \right\}.$$

**Proof.** (i) Let us introduce  $S_\alpha(x_0) := c.l.s.(X)$  and let  $E : K \rightarrow S_\alpha(x_0)$  denote the orthogonal projection onto  $S_\alpha(x_0)$ . We can define  $K_1 := S_\alpha(x_0)$ . Then the statement (i) follows directly from Lemma 1.3 and Theorem 2.3.

(ii) As before  $\Gamma_1 := E\Gamma$  and  $A_1 := AE = EAE$  are closely connected in  $E(K)$ . As  $A_1$  is bounded operator, and  $A_1$  and  $\Gamma_1$  are closely connected, we have

$$S_\alpha(x_0) = c.l.s. \{A^i \Gamma_1(H), i = 0, 1, \dots\} = c.l.s. \{(A - \alpha I)^i \Gamma_1(H), i = 0, 1, \dots\}.$$

Therefore, the last vector in the given Jordan chain,  $x_{l-1}$  must have a representation of the form

$$x_{l-1} = \Gamma_1 h_0 + (A - \alpha I)y, \quad ,$$

where  $y := \sum_{i=0}^{\infty} (A - \alpha I)^i \Gamma_1 h_{i+1} \in S_\alpha(x_0)$ .

Obviously,  $(A - \alpha I)^i \Gamma_1 h_{i+1} = 0$  for every  $i \geq l$ .

If  $y = 0$ , then  $x_{l-1} = \Gamma_1 h_0$ , which proves (ii). If  $y \neq 0$ , then it follows

$$x_0 = (A - \alpha I)^{l-1} x_{l-1} = (A - \alpha I)^{l-1} \Gamma_1 h_0.$$

Hence, we can take  $x_{l-1} = \Gamma_1 h_0 \in E\Gamma(H)$ .

**Remark 3.2** *Obviously, a typical situation is that  $\Gamma_1(H)$  is a proper subset of  $K_1$ . There exist examples of maximal Jordan chains that do not begin at  $\Gamma_1(H)$  (meaning  $x_{l-1} \notin \Gamma_1(H)$ ). Also, there are examples of chains that begin in  $\Gamma_1(H)$  that are not maximal. The meaning of the statement (ii) is that structure of the algebraic eigen-space  $S_\alpha(A)$  can be characterized by means of maximal non-degenerate Jordan chain with the last vectors  $x_{l-1} \in \Gamma_1(H)$ .*

**3.2.** Let  $\alpha \in R$  be a generalized pole not of positive type of  $Q \in N_\kappa(H)$ . We will focus on the decomposition within a single critical point. Therefore, it is not a loss of generality to assume that  $Q$  admits representation (1.3) and that  $\alpha \in R$  is the single critical point. For simplicity, we again use  $Q$ ,  $A$  and  $\Gamma$  rather than of  $Q_\alpha$ ,  $A_0$  and  $\Gamma_0$ .

Let  $K_0 \subseteq K$  be the Hilbert subspace that consists of all positive eigenvectors of the representing operator  $A$  at  $\alpha$ . Obviously,  $K_0$  is invariant subspace with respect to  $A$ . Let  $E_0 : K \rightarrow K_0$  be the orthogonal projection and  $\Gamma_0 := E_0 \Gamma$ . Then the Pontryagin

space  $(I - E_0)K$  is also invariant with respect to  $A$ , and operators  $\tilde{\Gamma} := (I - E_0)\Gamma$  and  $\tilde{A} := (I - E_0)A(I - E_0)$  are closely connected.

Now let  $x_0^1, \dots, x_{l_1-1}^1$  be a maximal non-degenerate Jordan chain of  $\tilde{A}$  at  $\alpha$  in the Pontryagin space  $(I - E_0)K$ . We define:  $E_1 : (I - E_0)K \rightarrow S_\alpha(x_0^1)$ ;  $K_1 := E_1(I - E_0)K$ . Then  $A_1 = AE_1$  and  $\Gamma_1 := E_1\Gamma$  are closely connected,  $\kappa_1$  is index of the Pontryagin space  $K_1$ . According to Proposition 3.1 (ii) we can consider  $x_{l_1-1}^1 = \Gamma_1 h_1$ . We continue that process until we exhaust all non-degenerate Jordan chains. Assume that there are  $r > 0$  such chains.

Let  $E := E_0 + E_1 + \dots + E_r$ . Then  $K = EK[+](I - E)K$ . Let us introduce  $E_{r+1} := I - E$ ,  $K_{r+1} := E_{r+1}K$ ,  $\Gamma_{r+1} = E_{r+1}\Gamma$ . Subspace  $K_{r+1}$  is invariant with respect to  $A$ . From the construction of the Pontryagin space  $K_{r+1}$  we conclude that all chains of  $A$  at  $\alpha$  that are contained in  $K_{r+1}$  are degenerate.

Using the above notation we can summarize the results in the following proposition.

**Proposition 3.3** *Let  $\alpha \in R$  be a generalized pole not of positive type of  $Q \in N_\kappa(H)$  given by minimal representation (1.1). Then*

- (i)  $K = K_0[+]K_1[+]\dots[+]K_r[+]K_{r+1}$  ,  
 where  $K_i$ ,  $i = 0, 1, \dots, r, r+1$  are Pontryagin spaces of indexes  $\kappa_i$ , respectively;  
 $\kappa_0 = 0, \kappa = \sum_{i=1}^{r+1} \kappa_i$ . With  $E_i$  denoting orthogonal projections onto  $K_i$ , operators  $A_i := AE_i$  and  $\Gamma_i := E_i\Gamma$ , are closely connected in  $K_i = E_iK$ ,  $i = 0, 1, 2, \dots, r+1$ . For  $i = 1, 2, \dots, r$ , operators  $A_i$  have single eigenvalue  $\alpha$  and respective non-degenerate Jordan chain  $x_0^i, \dots, x_{l_i-1}^i = \Gamma_i h_i$ . Hilbert space  $K_0$  consists of all positive eigenvectors of  $A$  at  $\alpha$ .
- (ii) For every  $i = 1, 2, \dots, r$  there exist  $h_i \in H$  such that subspace  $K_i$  is a linear span of the Jordan vectors

$$x_k^i = (A - \alpha)^{l_i-1-k} \Gamma_i h_i, \quad k = 0, \dots, l_i - 1.$$

- (iii)  $Q := Q_0 + Q_1 + \dots + Q_r + Q_{r+1}$  ,

$$Q_i(z) = \Gamma_i^+ (A_i - zE_i)^{-1} \Gamma_i \in N_{\kappa_i}(H), \quad i = 1, \dots, r.$$

Obviously, the decomposition obtained in the Proposition 3.3 is desirable and within  $\alpha$ .

## 4 Inverse of the function $Q(z) = \Gamma_0^+ (A - z)^{-1} \Gamma_0$

**Lemma 4.1** *Let operators  $\Gamma, \Gamma^+, J$  be as introduced in the Section 1. Assume also that  $\Gamma^+\Gamma$  is an boundedly invertible operator in the Hilbert space  $(H, (.,.))$ . Then for operator*

$$P := \Gamma (\Gamma^+\Gamma)^{-1} \Gamma^+$$

following statements hold:

- (i)  $P$  is orthogonal projection in Pontryagin space  $(K, [.,.])$ ;
- (ii) Scalar product does not degenerate on  $\Gamma(H)$  and therefore it does not degenerate on  $\Gamma(H)^{[\perp]} = \text{Ker} \Gamma^+$ .
- (iii)  $\text{Ker} \Gamma^+ = (I - P)K$
- (iv) Pontryagin space  $K$  can be decomposed as a direct orthogonal sum of Pontryagin spaces i.e.

$$K = (I - P) K[+] PK. \quad (4.1)$$

**Proof.** (i) Obviously  $P^2 = P$ .

According to well known properties of adjoint operators (see e.g. [JKL] p. 34) it is easy to verify  $\left[(\Gamma^+ \Gamma)^{-1}\right]^* = (\Gamma^+ \Gamma)^{-1}$  and then to verify  $[Px, y] = [x, Py]$ , i.e.  $P[*] = P$ . This proves (i).

(ii) If  $\Gamma h \neq 0$  and  $[\Gamma h, \Gamma g] = 0$ ,  $\forall g \in H$ , then  $(\Gamma^+ \Gamma h g) = 0$ ,  $\forall g \in H$ . This means  $\Gamma^+ \Gamma h = 0 \rightarrow h = 0 \rightarrow \Gamma h = 0$ . This is a contradiction that proves (ii).

(iii) It is sufficient to prove  $\text{Ker} \Gamma^+ = \text{Ker} P$ .

$$P := \Gamma (\Gamma^+ \Gamma)^{-1} \Gamma^+ \rightarrow \text{Ker} \Gamma^+ \subseteq \text{Ker} P.$$

Conversely, as  $\Gamma^+ \Gamma$  is boundedly invertible,

$$y \in \text{Ker} P \rightarrow 0 = \left[ \Gamma (\Gamma^+ \Gamma)^{-1} \Gamma^+ y, x \right] = \left( (\Gamma^+ \Gamma)^{-1} \Gamma^+ y, \Gamma^+ x \right), \forall \Gamma^+ x \in H.$$

$$R(\Gamma^+) = H \rightarrow (\Gamma^+ \Gamma)^{-1} \Gamma^+ y = 0 \rightarrow \Gamma^+ y = 0 \rightarrow y \in \text{Ker} \Gamma^+.$$

(iv) Note that it holds  $P\Gamma = \Gamma$  and  $\Gamma^+ P = \Gamma^+$ . Now the statement (iv) follows directly from (iii) and (ii).

If a function  $Q$  is given by (2.1) we define

$$P := \Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+, \quad (4.2)$$

$$\tilde{A} := (I - P) A (I - P) : (I - P) K \rightarrow (I - P) K$$

$$(\tilde{A} - z)^{-1} : (I - P) K \rightarrow (I - P) K.$$

Note

$$(I - P) (\tilde{A} - z)^{-1} (I - P) = \begin{pmatrix} (\tilde{A} - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

We prefer to use the notation on the left hand side, because it makes the following proofs shorter.

**Theorem 4.2** Assume that function  $Q \in N_\kappa(H)$  has the representation,

$$Q(z) = \Gamma_0^+ (A - z)^{-1} \Gamma_0 \quad (4.3)$$

where  $A$  is a self-adjoint bounded operator in a Pontryagin space  $K$  and  $\Gamma_0^+ \Gamma_0$  is boundedly invertible. Then the inverse function

$$\hat{Q}(z) := -Q(z)^{-1},$$

has the following representation

$$\hat{Q}(z) = (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ A(I - P) (\tilde{A} - z)^{-1} (I - P) A - (A - z) \right\} \Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1}. \quad (4.4)$$

Note that we did **not** assume here that  $Q$  satisfies minimality condition.

**Proof.** For projection  $P$  introduced by (4.2), according to Lemma 4.1 (iv), we have the following decomposition

$$A - zI = \begin{pmatrix} (I - P)(A - zI)(I - P) & (I - P)AP \\ PA(I - P) & P(A - zI)P \end{pmatrix}. \quad (4.5)$$

For  $z \in \rho(A)$  let us denote

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} := (A - z)^{-1}. \quad (4.6)$$

By solving operator equations derived from the identity

$$\begin{pmatrix} \tilde{A} - z(I - P) & (I - P)AP \\ PA(I - P) & P(A - zI)P \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I - P & 0 \\ 0 & P \end{pmatrix},$$

we get

$$W = \left\{ P(A - zI)P - PA(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)AP \right\}^{-1}. \quad (4.7)$$

We need not to find operators  $X$ ,  $Y$  and  $Z$ . We only need to understand what their domains and ranges are. Then from

$$\text{Ker}(\Gamma_0^+) = R(I - P), \quad R(\Gamma_0) = R(P)$$

we easily see

$$\Gamma_0^+ X \Gamma_0 = \Gamma_0^+ Y \Gamma_0 = \Gamma_0^+ Z \Gamma_0 = 0.$$

By substituting (4.6) into (4.3) and using (4.7) we get

$$Q(z) = \Gamma_0^+ \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix} \Gamma_0 = \Gamma_0^+ \left\{ P(A - zI)P - PA(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)AP \right\}^{-1} \Gamma_0. \quad (4.8)$$

Then, by substituting (4.8) and (4.4) into the following product and using definition of  $P$  we verify

$$\begin{aligned} Q(z) \hat{Q}(z) &= \\ &= \Gamma_0^+ \left\{ P(A - zI)P - PA(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)AP \right\}^{-1} \Gamma_0 \times \\ &(\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ A(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)A - (A - zI) \right\} \Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1} = \\ &= \Gamma_0^+ \left\{ P(A - zI)P - PA(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)AP \right\}^{-1} \times \\ &\left\{ PA(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)AP - P(A - zI)P \right\} \Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1} \\ &= \Gamma_0^+ (-P) \Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1} = -I. \end{aligned}$$

We will later use representation (4.4) to prove Theorem 5.1, a result about desirable decomposition. Let us first give some consequences of the representation (4.4).

**Corollary 4.3** *Let  $Q(z)$ ,  $\hat{Q}(z)$ ,  $\Gamma_0$ ,  $\Gamma_0^+$  be the same as in Theorem 4.2, then it holds*

$$\hat{Q}(z) \Gamma_0^+ = (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ -I + A(I - P) \left( \tilde{A} - z \right)^{-1} (I - P) \right\} (A - zI). \quad (4.9)$$

**Proof.** In the following derivations we will frequently use  $\Gamma_0^+ P = \Gamma_0^+$ ,  $P \Gamma_0 = \Gamma_0$ .

$$\begin{aligned} \hat{Q}(z) \Gamma_0^+ &= (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ A(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)A - (A - zI) \right\} \Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \\ &= (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ A(I - P) \left( \tilde{A} - z \right)^{-1} (I - P)(A - zI)P - (A - zI)P \right\} \end{aligned}$$

$$\begin{aligned}
&= (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \times \\
&\left\{ A(I-P) \left( \tilde{A} - z \right)^{-1} (I-P) (A-zI) (P-I) + A(I-P) \left( \tilde{A} - z \right)^{-1} (I-P) (A-zI) - (A-zI) P \right\} \\
&= (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ -A(I-P) + A(I-P) \left( \tilde{A} - z \right)^{-1} (I-P) (A-zI) - (A-zI) P \right\} \\
&= (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ -(A-zI) + A(I-P) \left( \tilde{A} - z \right)^{-1} (I-P) (A-zI) \right\} \\
&= (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ \left\{ -I + A(I-P) \left( \tilde{A} - z \right)^{-1} (I-P) \right\} (A-zI).
\end{aligned}$$

**Corollary 4.4** *Let again  $Q(z)$ ,  $\hat{Q}(z)$ ,  $\Gamma_0$ ,  $\Gamma_0^+$  be the same as in Theorem 4.2 Then the inverse  $\hat{Q}$  has representation (1.4), i.e.*

$$\hat{Q}(z) = -Q(\bar{z}_0)^{-1} + (z - \bar{z}_0) \hat{\Gamma}^+ \left( (I + (z - z_0) (\hat{A} - z)^{-1}) \hat{\Gamma} \right),$$

where  $\hat{A}$  is a self-adjoint linear **relation** with critical eigenvalue at  $\infty$ , i.e. it holds

$$\hat{A}(0) = R(P) = R(\Gamma_0). \quad (4.10)$$

**Proof.** The function  $Q \in N_\kappa(H)$  that admits representation (4.3) is a special case of the function that admits representation (1.1). Let  $z_0 \in \rho(A)$  and let us introduce  $\Gamma$  by

$$\Gamma := (A - z_0)^{-1} \Gamma_0. \quad (4.11)$$

Then from (4.3) it easily follows (1.1) where  $Q(z_0)^* = \Gamma_0^+ (A - \bar{z}_0) \Gamma_0$ .

According to Lemma 1.4, the inverse  $\hat{Q}$  admits representation (1.4). Then for  $z = z_0$ , from (1.5) we get  $\Gamma_{z_0} = \Gamma$ , and from (1.6) we get

$$(\hat{A} - z_0)^{-1} = (A - z_0)^{-1} - \Gamma_{z_0} Q(z_0)^{-1} \Gamma_{z_0}^+ \quad (4.12)$$

From (4.11), it follows

$$\Gamma_{z_0} = (A - z_0)^{-1} \Gamma_0 \text{ and } \Gamma_{z_0}^+ = \Gamma_0^+ (A - z_0)^{-1}.$$

Substituting this into (4.12) gives

$$\begin{aligned}
(\hat{A} - z_0)^{-1} &= (A - z_0)^{-1} - (A - z_0)^{-1} \Gamma_0 Q(z_0)^{-1} \Gamma_0^+ (A - z_0)^{-1} \\
&= (A - z_0)^{-1} \left( I - \Gamma_0 Q(z_0)^{-1} \Gamma_0^+ (A - z_0)^{-1} \right).
\end{aligned}$$

According to the Corollary 4.3 we get

$$\begin{aligned}
(\hat{A} - z_0)^{-1} &= (A - z_0)^{-1} \left( I + P \left( -I + A(I-P) \left( \tilde{A} - z_0 \right)^{-1} (I-P) \right) \right) \\
&= (A - z_0)^{-1} \left( I - P + PA(I-P) \left( \tilde{A} - z_0 \right)^{-1} (I-P) \right) \\
&= (A - z_0)^{-1} \left( I + PA(I-P) \left( \tilde{A} - z_0 \right)^{-1} \right) (I-P)
\end{aligned}$$

From this we conclude  $\text{Ker} \left( \hat{A} - z_0 \right)^{-1} \supseteq R(P)$  and, therefore  $\hat{A}(0) \supseteq R(\Gamma_0)$ .

In order to prove,  $\text{Ker} \left( \hat{A} - z_0 \right)^{-1} \subseteq R(\Gamma_0)$ , assume to the contrary that there exist  $0 \neq (I - P)y \in \text{Ker} \left( \hat{A} - z_0 \right)^{-1}$ . Then from

$$\left( \hat{A} - z_0 \right)^{-1} = (A - z_0)^{-1} \left( I + PA(I - P) \left( \tilde{A} - z_0 \right)^{-1} \right) (I - P) \quad (4.13)$$

and from the fact that  $z_0$  is a regular point of the operator  $A$  we get

$$\left( I + PA(I - P) \left( \tilde{A} - z_0 \right)^{-1} \right) (I - P)y = 0$$

Let us define:  $(I - P)g := \left( \tilde{A} - z_0 \right)^{-1} (I - P)y \neq 0$

Then we have

$$\left( \tilde{A} - z_0 \right) (I - P)g + PA(I - P)g = 0$$

Having in mind  $\tilde{A} = (I - P)A(I - P)$  we get

$$\begin{aligned} ((I - P)A - z_0(I - P) + PA)(I - P)g &= 0 \\ (A - z_0)(I - P)g &= 0 \end{aligned}$$

This means that  $(I - P)g \neq 0$  is an eigenvector of  $A$  in the eigenvalue  $z_0$ , which is a contradiction. Therefore,  $\text{Ker} \left( \hat{A} - z_0 \right)^{-1} = R(\Gamma_0)$ , which proves the statement.

**Remark 4.5** *One consequence of the Corollary 4.4 is that the inverse  $\hat{Q}$  of  $Q$  cannot have operator representation of the form (1.1);  $\hat{A}$  has to be a linear relation. Therefore, operator representation (4.4) is essentially a new representation.*

**Remark 4.6** *If  $Q$  admits representation (4.3) then a pole cancellation function can be conveniently defined by*

$$\eta(z) := Q(z)^{-1} \Gamma_0^+ \left\{ x_0 + (z - \alpha)x_1 + \dots + (z - \alpha)^{l-1}x_{l-1} \right\} \quad , \quad (4.14)$$

where  $\{x_0, x_1, \dots, x_{l-1}\}$  is a Jordan chain of  $A$  at  $\alpha$ .

That is how the expression for  $\hat{Q}(z)\Gamma_0^+$ , proven in Corollary 4.3, comes into play in the study of pole cancellation functions.

Pole cancellation functions of the form (4.14) were constructed in [1] for the functions  $Q \in N_{\kappa}^{n \times n}$  and were used there to characterize regular poles including their multiplicities. Existence of generalized poles was characterized in [2], without characterization of their multiplicities. Much later, in [3], the functions of the form (4.14) were used to characterize generalized poles of the function  $Q \in N_{\kappa}(H)$ , including their multiplicities.

Note, if a Jordan chain of  $A$  at  $\alpha$  of length  $l$  satisfies  $x_{l-1} = \Gamma_0 h$ , then, according to Corollary 4.3 the pole cancellation function (4.14) has a very simple form  $\eta(z) = -(z - \alpha)^l h$ .

## 5 A desirable decomposition of the function $\hat{Q}$

**Theorem 5.1** *Let  $Q(z) = \Gamma_0^+ (A - z)^{-1} \Gamma_0 \in N_{\kappa}(H)$ , where  $A$  is a bounded self-adjoint operator and let  $\alpha \in R$  be a generalized pole of  $Q$ . Assume that the derivative  $Q'(\infty) := \lim_{z \rightarrow \infty} zQ(z) = -\Gamma_0^+ \Gamma_0$  is boundedly invertible operator. Then the inverse  $\hat{Q}(z) := -Q(z)^{-1}$  has a desirable decomposition*

$$\hat{Q}(z) = \hat{Q}_1(z) + \hat{Q}_2(z); \quad \hat{Q}_1 \in N_{\hat{\kappa}_1}(H), \quad \hat{Q}_2 \in N_{\hat{\kappa}_2}(H); \quad \hat{\kappa}_1 + \hat{\kappa}_2 = \kappa.$$

That decomposition has the following properties:

- (i) Function  $\hat{Q}_1$  may have a generalized zero at  $\alpha$  and cannot have any generalized pole in  $C$ .
- (ii) The negative index  $\hat{\kappa}_1$  is equal to the number of negative eigenvalues of the self-adjoint operator  $(\Gamma_0^+ \Gamma_0)^{-1}$  in the Hilbert space  $H$ .
- (iii) If  $Q$  has a generalized zero at  $\alpha$  then the function  $\hat{Q}_2$  has a generalized pole at  $\alpha$ .
- (iv) The function  $\hat{Q}_2$  has minimal representation  $\hat{Q}_2(z) := \tilde{\Gamma}^+ (\tilde{A} - z)^{-1} \tilde{\Gamma}$ ,

where  $\tilde{\Gamma} = (I - P)A\Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1}$ ,  $\tilde{A} = (I - P)A(I - P)$ .

**Proof.** From (4.4), it follows  $\hat{Q} = \hat{Q}_1 + \hat{Q}_2$  where

$$\begin{aligned}\hat{Q}_1(z) &= -(\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ (A - z) \Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1} \\ \hat{Q}_2(z) &:= (\Gamma_0^+ \Gamma_0)^{-1} \Gamma_0^+ A(I - P) (\tilde{A} - z)^{-1} (I - P)A\Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1}\end{aligned}$$

Statements (i) and (iii) follow immediately. Let us prove the remaining statements.

We know  $\hat{Q} \in N_\kappa(H)$  and  $\hat{\kappa}_1 + \hat{\kappa}_2 \geq \kappa$ . Let us denote by  $\kappa'$  and  $\kappa''$  negative indexes of  $PK$  and  $(I - P)K$ , respectively. Then, according to (4.1)  $\kappa' + \kappa'' = \kappa$ .

For  $f, g \in H$  we have

$$\left( \frac{\hat{Q}_1(z) - \hat{Q}_1(w)^*}{z - \bar{w}} f, g \right) = \left( (\Gamma_0^+ \Gamma_0)^{-1} f, g \right)$$

As  $(\Gamma_0^+ \Gamma_0)^{-1}$  is bounded, hence defined on the whole  $H$ , we can consider  $f = \Gamma_0^+ \Gamma_0 f_0$  and  $g = \Gamma_0^+ \Gamma_0 g_0$ , where  $f_0$  and  $g_0$  run through entire  $H$  when  $f, g$  run through entire  $H$ . Therefore

$$\left( (\Gamma_0^+ \Gamma_0)^{-1} f, g \right) = (f_0, \Gamma_0^+ \Gamma_0 g_0) = [\Gamma_0 f_0, \Gamma_0 g_0]$$

As  $R(\Gamma_0) = R(P)$  we conclude that  $\hat{\kappa}_1 = \kappa'$ . Therefore,  $\kappa' + \hat{\kappa}_2 \geq \kappa = \kappa' + \kappa''$ , and  $\hat{\kappa}_2 \geq \kappa''$ .

If we introduce  $\tilde{\Gamma} = (I - P)A\Gamma_0 (\Gamma_0^+ \Gamma_0)^{-1}$ , then  $\hat{Q}_2(z) := \tilde{\Gamma}^+ (\tilde{A} - z)^{-1} \tilde{\Gamma}$ , where  $\tilde{A}$  is a self-adjoint operator in  $(I - P)K$ . That means  $\hat{\kappa}_2 \leq \kappa''$ , i.e.  $\hat{\kappa}_2 = \kappa''$ .

This proves the remaining statements of the theorem, including  $\hat{\kappa}_1 + \hat{\kappa}_2 = \kappa$ .

**Corollary 5.2** *Let again  $Q \in N_\kappa(H)$  be as in Theorem 4.2 and let us consider representation (1.1) of  $\hat{Q}$ . Let  $(\hat{A}, \hat{\Gamma}, \hat{K})$  be a corresponding triplet. Then the linear relation  $\hat{A}$  cannot be an operator. There exists an invariant subspace  $\hat{K}_1 \subseteq \hat{K}$  of the linear relation  $\hat{A}$ , where the negative index  $\hat{\kappa}_1$  of  $\hat{K}_1$  is equal to the number of negative eigenvalues of  $(\Gamma_0^+ \Gamma_0)^{-1}$ .*

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